

Module 4

Section 1: Inflation

The following example illustrates how inflation affects the purchasing power of money. Suppose we have \$100 now and a gallon of milk currently costs \$4, but instead of buying 25 gallons of milk today, we invest the \$100 for 2 years at an annual effective rate of interest of $i = 8\%$. After 2 years we have $100(1.08)^2 = 116.64$. However, the price of milk may no longer be \$4. Generally commodity prices will increase over time, and we measure such increases by the **inflation rate**. For example, if the annual inflation rate over the 2-year period is $r = 5\%$, then the cost of a gallon of milk after two years is $4(1.05)^2 = 4.41$. Then at the end of two years, we can purchase $116.64/4.41$ (about 26.45) gallons of milk.

The 8% rate in the previous paragraph is called the **nominal interest rate**. The example illustrates that using a nominal annual interest rate of 8% and an annual inflation rate of 5%, then 25 today is equivalent to about 26.45 in two years. The annual effective interest rate implied by this equivalence is called the **real rate of return** and is denoted i' . For this example, we have

$$25(1 + i')^2 = \frac{116.64}{4.41}$$

Note where all these numeric values came from; i.e. $25 = \frac{\$100}{\$4 \text{ per gallon}}$,

$116.64 = \$100(1.08)^2$, and $4.41 = \$4(1.05)^2$ per gallon. Substituting into the above equation, we get

$$\frac{100(1 + i')^2}{4} = \frac{100(1.08)^2}{4(1.05)^2} = \frac{100(1 + i)^2}{4(1 + r)^2}$$

Cancelling off common factors and taking square roots, this example illustrates the main fact relating the nominal annual rate i , the annual inflation rate r , and the annual real rate of return i' ; namely,

$$1 + i = (1 + r)(1 + i')$$

Module 4 Section 1 Problems:

1. Using a nominal rate, an initial investment of 1000 accumulates to 1071 after 1 year. Assuming an annual inflation rate of 2%, determine the accumulated value of 1000 after 1 year using the annual effective real rate of return.
2. In order to save for retirement, a 25-year old begins depositing 250 at the beginning of each month, beginning on the 25th birthday. Deposits continue until age 65, with the last deposit one month before the 65th birthday.
 - (a) Using a nominal interest rate of 9% compounded monthly, determine the accumulated value on the 65th birthday.
 - (b) Assuming the same nominal rate of 9% compounded monthly in part (a), and assuming an inflation rate of 3% compounded monthly, determine the accumulated value on the 65th birthday using the real rate of return.
 - (c) Redo part (b) except using an inflation assumption of 4% compounded monthly for the first 20 years and 2% compounded monthly thereafter.
3. The present value of a 30-year annuity immediate with semiannual payments of 1000 is 27,675.56, using an annual real rate of return of i . If the assumed inflation rate is 2% compounded semiannually, determine the implied nominal annual interest rate, compounded semiannually.

Solutions to Module 4 Section 1 Problems:

1. We have $1000(1 + i) = 1071 \Rightarrow 1 + i = 1.071$, where i is the nominal annual effective rate. Since the annual effective inflation rate is $r = .02$, then the annual effective real rate of return, i' , is calculated as follows:

$$(1 + i')(1.02) = 1.071 \Rightarrow i' = .05$$

Therefore, 1000 accumulates to $1000(1.05) = 1050$ using the annual effective real rate of return.

2. There are $12(40) = 480$ payments of 250 each. The valuation date is one period (month) after the last payment, and so the value of the annuity is $AV = 250\ddot{s}_{\overline{480}|}$. We must determine the meir to use for the calculation.

(a) We use an meir of $i = \frac{9\%}{12} = 0.75\%$, obtaining $AV = 1,179,108$.

(b) The monthly effective nominal rate is $i = 0.0075$ and the monthly effective inflation rate is $r = \frac{.03}{12} = .0025$. Therefore the monthly effective real rate of return is $i' = \frac{1.0075}{1.0025} - 1$. Using this meir for the calculation, we have $AV = 498,329$.

(c) As in part (b), the monthly effective real rate of return is $i' = \frac{1.0075}{1 + \frac{.04}{12}} - 1 = i_1$ for the first 240 months, and $i' = \frac{1.0075}{1 + \frac{.02}{12}} - 1 = i_2$ thereafter. Therefore we have $AV = 250\ddot{s}_{\overline{240}|i_1}(1 + i_2)^{240} + 250\ddot{s}_{\overline{240}|i_2} = 545,782$.

3. There are a couple of things to be careful of in this problem. First, notice the problem uses the symbol i for the real rate of return, whereas we use i' . Also be careful with all the different interest rate types within the problem. Some are nominal rates and others are effective rates. We proceed as follows:

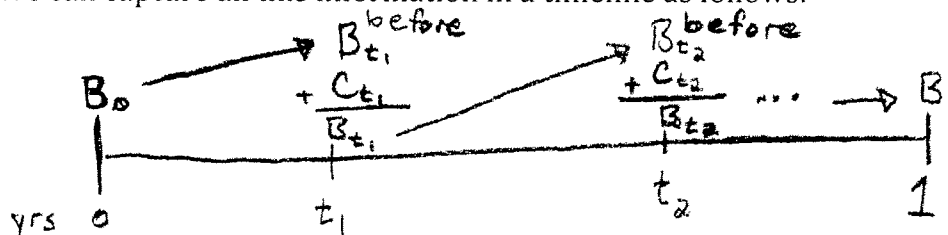
Since the seir for which $27675.56 = 1000a_{\overline{60}|}$ is 3%, the semiannual effective real rate of return is $i' = .03$. The semiannual effective inflation rate is $r = .01$. The implied semiannual effective nominal interest rate is determined as follows: $1 + i = (1 + i')(1 + r) = (1.03)(1.01) = 1.0403 \Rightarrow i = .0403$. Therefore, the implied nominal annual interest rate, compounded semiannually, is 8.06%.

Section 2: Dollar and Time Weighted Rates

Dollar Weighted and Time Weighted Interest Rates are used to evaluate investor performance and investment fund managers' performance.

Set-up: We have a fund account value, or balance, at the beginning of the year, denoted by B_0 , a balance at the end of the year, denoted by B_1 , and transactions during the year. These transactions during the year, at time t ($0 < t < 1$), are deposits and/or withdrawals denoted by D_t and W_t , respectively. The net contribution for the transaction at time t is $C_t = D_t - W_t$. Of course this may be negative. At each transaction date, there are balances immediately before and immediately after the transaction, that we denote B_t^{before} and B_t , respectively.

We can capture all this information in a timeline as follows:



Dollar-Weighted Return: This return is the constant simple interest rate, i , at which the beginning balance and net contributions are equivalent to the ending balance, using an end of year valuation date. Note that the accumulated value at time $t = 1$ of the beginning balance is $B_0(1 + i)$, and by definition of dollar-weighted return, the accumulated value at time $t = 1$ of the net contribution at time t is $C_t(1 + i(1 - t))$. Therefore, $B_1 = B_0(1 + i) + \sum C_t(1 + i(1 - t))$. Solving this equation for i , we obtain

$$i_{DW} = \frac{B_1 - B_0 - \sum C_t}{B_0 + \sum C_t(1 - t)} = \frac{I}{Exp}$$

where I is the amount of interest earned during the year, and Exp is the *exposure*, which is the amount of the fund "exposed to interest" during the year.

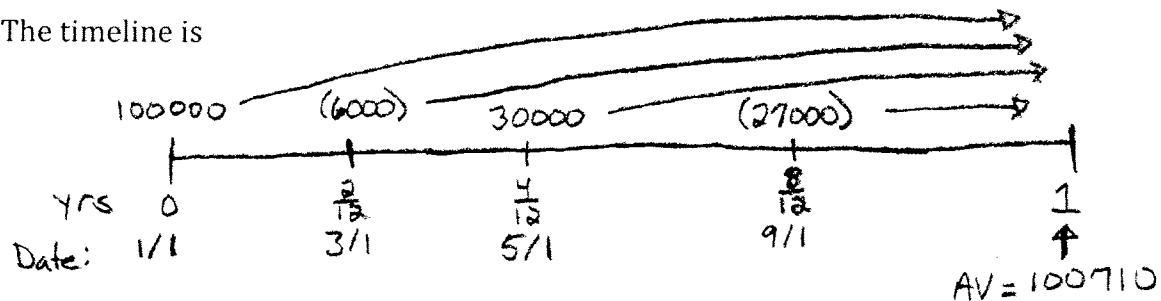
Time-Weighted Return: This return is determined by compounding. We calculate the time-weighted return by setting the annual time-weighted accumulation factor, $1 + i_{tw}$, equal to the product of the periodic accumulation factors, where the periods are determined by the transaction dates. For example, the first periodic accumulation factor is $\frac{B_{t_1}^{before}}{B_0}$, the second periodic accumulation factor is $\frac{B_{t_2}^{before}}{B_{t_1}}$, and so on. Then $1 + i_{tw}$ equals the product of these accumulation factors. The notation makes the calculation of the time-weighted return look complicated, but it's actually easier than the calculation of the dollar-weighted return.

Module 4 Section 2 Problems:

1. At the beginning of the year, an account has a balance of 100,000. There is a deposit of 30,000 on May 1, and there are withdrawals of 6,000 and 27,000 on March 1, and September 1, respectively. There are no other deposits or withdrawals. The account balance at the end of the year is 100,710. Determine the dollar weighted return.
2. Investor A and Investor B each have 1000 in the same investment account on January 1. On April 1, Investor A deposits an additional 900 whereas Investor B withdraws 100. The balance in each account immediately before the transactions is 1100. On October 1, Investor A withdraws 500 and Investor B deposits 750. The balance in Investor A's account immediately before the withdrawal is 2500, whereas the balance in Investor B's account immediately before the deposit is 1250. On December 31, the balance in each account is 1600.
 - (a) Determine the dollar-weighted rate of return in Investor A's account.
 - (b) Determine the dollar-weighted rate of return in Investor B's account.
 - (c) Determine the time-weighted rate of return in Investor A's account.
 - (d) Determine the time-weighted rate of return in Investor B's account.
3. An investor has an account with a beginning balance on January 1 of 10,000. The investor makes deposits of 300 at the end of each month. There are no withdrawals from the account. If the account balance is 14473.75 on the following January 1, determine the dollar weighted return the investor received.
4. An account has a beginning of year balance of 5000. On June 20th there is a deposit of 500. There are no other deposits or withdrawals. The end of year balance is 5768 and the time weighted return for the year is 5.06%. Determine the balance in the account on June 20th immediately after the deposit on that date.
5. An account has a dollar weighted return of 10% during the year. The beginning of year balance is 2000. There are no deposits and there is only one withdrawal of 300 during the year. The ending balance is 1876.25. Determine the date of the withdrawal.

Solutions to Module 4 Section 2 Problems:

1. The timeline is



We have:

$$100710 = 100000(1 + i) - 6000\left(1 + \frac{10}{12}i\right) + 30000\left(1 + \frac{8}{12}i\right) - 27000\left(1 + \frac{4}{12}i\right)$$

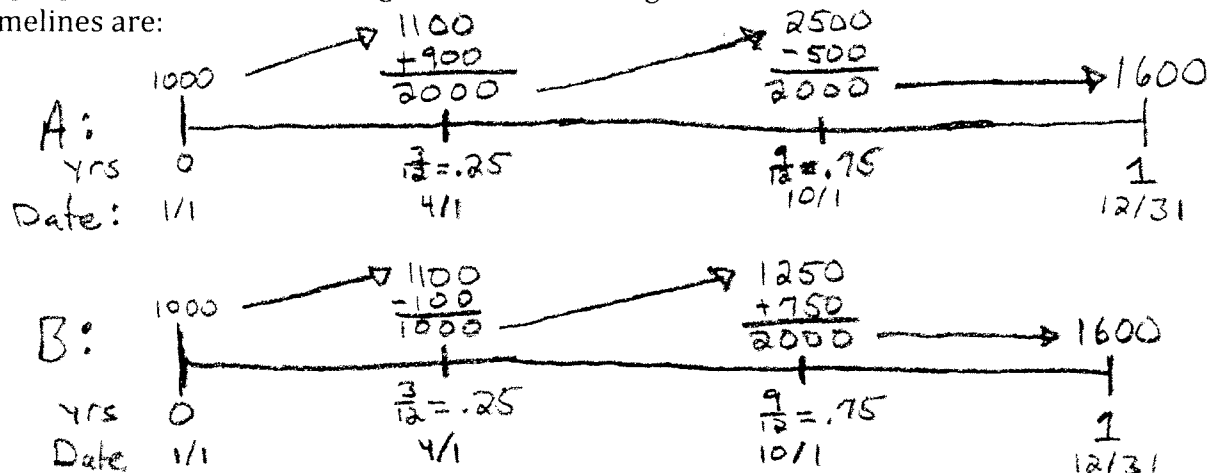
This is a linear equation in i . The details of the solution are provided so that we can see the resulting formula. Distribute the dollar amounts, move all terms without an i to the other side, factor out i , and divide. We have

$$i = \frac{100710 - 100000 + 6000 - 30000 + 27000}{100000 - 6000 \cdot \frac{10}{12} + 30000 \cdot \frac{8}{12} - 27000 \cdot \frac{4}{12}} = \frac{3710}{106000} = 0.035$$

Notice the beginning balance is $B_0 = 100000$, the total amount of contributions into the account during the year is $C = -6000 + 30000 - 27000 = -3000$, and the ending balance is $B_1 = 100710$. Therefore the amount of interest earned during the year must be $I = 100710 - 100000 - (-3000) = 3710$. This is the numerator of the next to last expression in the string of equations above. The denominator of that expression is the exposure. Notice that the beginning balance is "exposed" during the entire year and so the full 100000 contributes to the exposure amount. The first withdrawal of 6000 is made at time $t = \frac{2}{12}$. It is "exposed" for $\frac{10}{12}$ of the year and, since it's a withdrawal, it contributes $(-6000 \cdot \frac{10}{12})$ to the exposure amount. Continue and we get the denominator. Therefore,

$$i = \frac{I}{Exp} = \frac{3710}{106000} = 0.035$$

2. This problem illustrates the reasoning behind the names and also when it is appropriate to use dollar-weighted and time-weighted rates of returns. The timelines are:



(a) Investor A makes contributions during the year totaling $900 - 500 = 400$. Therefore Investor A earns $1600 - 1000 - 400 = 200$ in interest during the year. The exposure during the year is $1000 + 900(.75) - 500(.25) = 1550$, and so the dollar-weighted return for Investor A is $i_{DW}^A = \frac{200}{1550} \approx 0.12903$.

(b) Investor B makes contributions during the year totaling $750 - 100 = 650$. Therefore Investor B earns $1600 - 1000 - 650 = -50$ in interest during the year. The exposure during the year is $1000 - 100(.75) + 750(.25) = 1112.50$, and so the dollar-weighted return for Investor B is $i_{DW}^B = \frac{-50}{1112.50} \approx -0.04494$.

REMARKS:

1. The dollar-weighted rate of return is a measure of *investor* performance. It will depend on the dollar amount of each transaction (thus *dollar-weighted*) as well as when each transaction is made, which are all investor decisions.
2. Investor A and Investor B had the same starting value of 1000 and the same ending value of 1600. However, Investor A made contributions totaling 400 whereas Investor B made contributions totaling 650. Therefore, it should be obvious to us that Investor A has a better dollar-weighted return during the year than Investor B.

(c) For Investor A we have $1 + i_{TW}^A = \frac{1100}{1000} \cdot \frac{2500}{2000} \cdot \frac{1600}{2000} \Rightarrow i_{TW}^A = 0.10$.

(d) For Investor B we have $1 + i_{TW}^B = \frac{1100}{1000} \cdot \frac{1250}{1000} \cdot \frac{1600}{2000} \Rightarrow i_{TW}^B = 0.10$.

REMARKS:

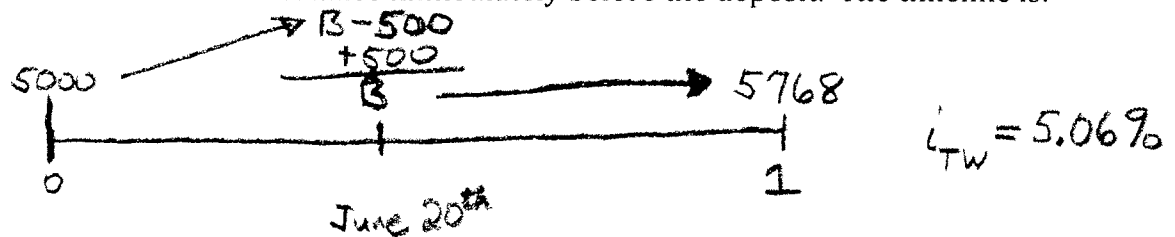
1. The time-weighted rate of return is a measure of *fund*, or *investment manager* performance. It depends only on the periodic effective returns between the times of the transactions (thus *time-weighted*) and not on the dollar amount of the transactions. This measure will not penalize (or reward) the fund manager based on an investor's deposit and withdrawal decisions.
2. Per the previous remark, it should be obvious to us that the time-weighted returns for Investor A and Investor B are equal, since they are both invested in the same investment account.

3. $I = 14473.75 - 10000 - 300(12) = 873.75$ is the amount of interest earned during the year. The exposure is $Exp = 10000 + 300\left(\frac{11}{12}\right) + 300\left(\frac{10}{12}\right) + \dots + 300\left(\frac{1}{12}\right)$. Notice that the last deposit of 300 does not contribute to the exposure since it is made at the end of the year and earns no interest. We can factor out $\frac{300}{12}$ from each term in the last 11 terms of the above expression for the exposure. Using the basic arithmetic sum formula

$$1 + 2 + \dots + n = \frac{n \cdot (n + 1)}{2}$$

we get $Exp = 10000 + \frac{300}{12} \cdot \frac{11 \cdot 12}{2} = 11650$. Therefore, $i_{DW} = \frac{873.75}{11650} = 0.075$.

4. Let B denote the account balance immediately after the June 20th deposit of 500. Then $B - 500$ is the account balance immediately before the deposit. The timeline is:



The actual time value associated to June 20th is not relevant, only the periodic accumulation factors during the periods from the beginning of the year to June 20th, and then from June 20th to the end of the year. These accumulation factors are $\frac{B-500}{5000}$ and $\frac{5768}{B}$, respectively. Since $i_{TW} = 0.0506$ the annual accumulation factor is 1.0506. Therefore, $1.0506 = \frac{B-500}{5000} \cdot \frac{5768}{B} \Rightarrow B = 5600$.

5. $I = 1876.25 - 2000 - (-300) = 176.25$ is the amount of interest earned during the year. Let t denote the time, in years, of the deposit. The exposure is

$$Exp = 2000 - 300(1 - t) = 1700 + 300t$$

Since $i_{DW} = 0.1$, we have

$$0.1 = \frac{I}{Exp} = \frac{176.25}{1700 + 300t} \Rightarrow t = 0.20833333 \text{ years}$$

Multiplying by 12, we see that $t = 2.5$ months. The withdrawal is made 2.5 months into the year; i.e. the withdrawal is made on March 15.

Section 3: Portfolio and Investment Year Methods

A quality of human behavior that many investors possess is that they will search for the highest return for their initial investment, but will not actively manage their investment by continuing to search for the highest return thereafter. Banks and other financial institutions that compete for the deposits of investors use **investment year** and **portfolio** interest rates as a means to try to capitalize on this behavior.

Investment year interest rates are “new money” rates that are typically higher than ongoing portfolio rates and are used to entice investors to deposit their money with the financial institution advertising the rates. For example, for new deposits at the beginning of 2010, an institution may have offered an interest rate of 6% (this is an *investment year* rate), whereas for other deposits (say money that was initially deposited in 2006), they may only be paying a 2010 interest rate of 5% (this is a *portfolio* rate). We usually capture investment year and portfolio rates in a table. The following table illustrates a 2-year investment year/portfolio interest rate table.

Year	Investment Year Rates		Portfolio Rates
Y	i_1^Y	i_2^Y	i^{Y+2}
2008	5.5%	5.5%	5.0%
2009	6.0%	5.5%	4.0%
2010	6.0%	5.0%	3.0%
2011	5.5%	4.5%	3.0%

We read the table across and then down. For example, corresponding to the row for year 2009, reading across and then down, we have $i_1^{2009} = 6.0\%$, $i_2^{2009} = 5.5\%$, $i^{2011} = 4.0\%$, $i^{2012} = 3.0\%$, and $i^{2013} = 3.0\%$. This means that the new money rate for deposits made at the beginning of 2009 is 6% for the first year (2009) and 5.5% for the second year (2010). Then the new money reverts back to the portfolio and earns the portfolio rates of 4% for 2011 and 3% for years 2012 and 2013.

Module 4 Section 3 Problems:

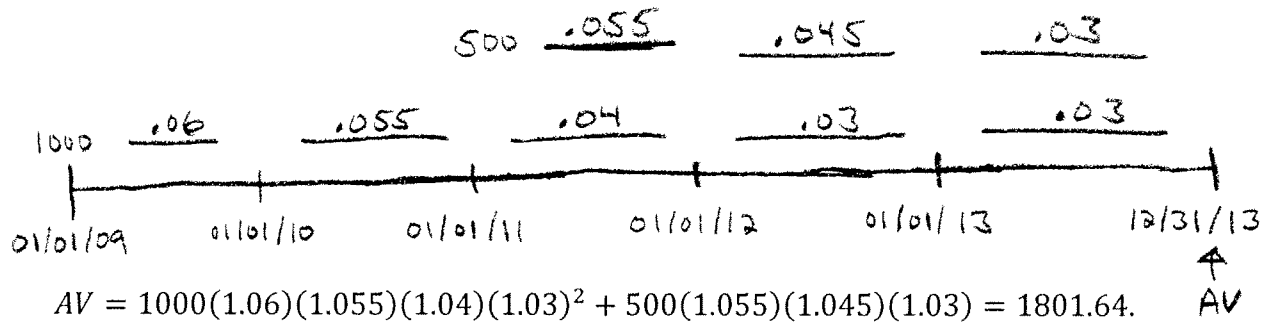
For each problem below, use the 2-year investment year/portfolio interest rate table from the previous page, repeated here for your convenience.

Year	Investment Year Rates		Portfolio Rates
Y	i_1^y	i_2^y	i^{Y+2}
2008	5.5%	5.5%	5.0%
2009	6.0%	5.5%	4.0%
2010	6.0%	5.0%	3.0%
2011	5.5%	4.5%	3.0%

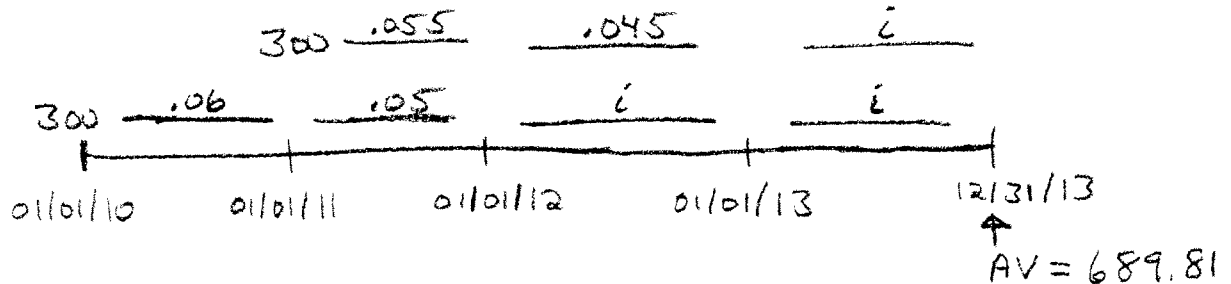
1. Determine the total accumulated value on 12/31/2013 if 1000 is deposited on 1/1/2009 and another 500 is deposited on 1/1/2011.
2. Unfortunately your incompetent coworker was charged with inputting the investment year rates and portfolio rates in the above table, and there is a mistake. Although it is true that portfolio rates for 2012 and 2013 are equal, the value for these rates is strictly between 1% and 3%. It is known that a particular investor deposited 300 at the beginning of 2010 and another 300 at the beginning of 2011 and on 12/31/2013 had an account balance of 689.81. Determine the 2012 (and 2013) portfolio rate.

Solutions to Module 4 Section 3 Problems:

1. We can capture the rates each year in a timeline as follows:



2. Again, we capture the rates each year in a timeline as follows:



We have $689.81 = 300(1.06)(1.05)(1+i)^2 + 300(1.055)(1.045)(1+i)$. This is a quadratic equation in $1+i$. Using the quadratic formula with

$$\begin{aligned} a &= 300(1.06)(1.05) \\ b &= 300(1.055)(1.045) \\ c &= -689.81 \end{aligned}$$

The minus sign in front of the radical gives an extraneous solution.

Section 4: Term Structure of Interest Rates, et al.

IDEA: Not only are short-term and long-term interest rates generally different at any point in time, but they also change over time. This phenomenon is called the **term structure of interest rates**.

The following table is a hypothetical table illustrating the term structure of interest rates. We can extend the values in the table to a continuous graph, and the resulting graphical illustration is called the **yield curve** corresponding to the table. The interest rates in the table are called **spot rates**; these are today's rates.

Hypothetical Term Structure of Interest Rates	
Length of Investment	Interest Rate
1 year	7.00%
2 years	8.00%
3 years	8.75%
4 years	9.25%
5 years	9.50%

The notation we use for spot rates is s_1 for the 1-year spot rate (7% in the table above), s_2 for the 2-year spot rate (8% above), etc. The following example illustrates how we use spot rates from yield curves.

If person A invests 100 for 2 years and person B invests 100 for 3 years, both using the corresponding spot rates in the table above, then A would have $100(1.08)^2 = 116.64$ at the end of 2 years, whereas B would have $100(1.0875)^3 = 128.61$ at the end of 3 years.

Since both A and B started with the same amount, a natural question to ask is "What annual effective interest rate must A receive between year 2 and year 3 so that A has the same amount at the end of year 3 as B does?" It should be clear that the initial amount of 100 is irrelevant, and we solve $(1.0875)^3 = (1.08)^2(1 + i)$. This i is called the **forward rate** from time 2 to time 3, and I'll denote it by $f_{[2,3]}$.

Unless told otherwise, all the spot rates and forward rates are annual effective. Similar to the last paragraph, if $k < n$ then we can relate the k -year and n -year spot rates to the forward rate from time k to time n as follows:

$$(1 + s_n)^n = (1 + s_k)^k (1 + f_{[k,n]})^{n-k}$$

Module 4 Section 4 Problems:

1. Given a two-year spot rate of 4% and a five-year spot rate of 5%, determine the annual forward rate from time 2 to time 5.
2. Suppose the current 1-year spot rate is 3% and the forward rate from time 1 to time 2 consistent with the current term structure of interest rates is 2%. Determine the 2-year discount factor from time 2 back to time 0.
3. You are given 1-year and 2-year spot rates of 4% and 5%, respectively, and a forward rate from time 2 to time 3 of 8%.

(a) Determine the price of a 1000 face value 3-year bond with 3% annual coupons, redeemable at par, that is consistent with this term structure of interest rates.

(b) Determine the corresponding annual yield that is consistent with this term structure of interest rates for 3-year 3% annual coupon bonds. [Note: The face value is not given, and in fact, we will show that the annual yield is independent of the face value. Therefore you may use any face value you want. Also since we cannot proceed otherwise, assume the bond is redeemable at par.]

4. The annual yield rate on zero coupon bonds with duration k years is given by

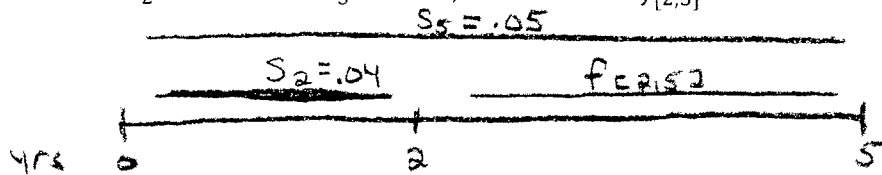
$$i_k = 0.03 + 0.005k, \quad k = 1, 2, 3$$

Determine the annual forward rate from time 1 to time 3 that is consistent with these yields.

Note: A zero-coupon bond is just that; a bond that has no coupons and only pays the redemption value at maturity. For a k -year zero-coupon bond, an investor pays the price for the bond today in return for the redemption value in k years. Therefore, the yield rate on a k -year zero-coupon is precisely the k -year spot rate.

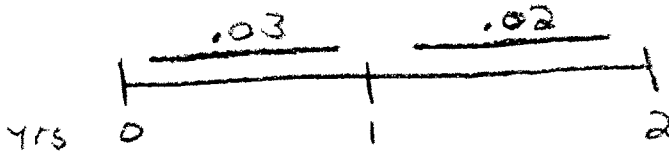
Solutions to Module 4 Section 4 Problems:

1. We have $s_2 = 0.04$ and $s_5 = 0.05$, and we seek $f_{[2,5]}$. The timeline is:



We have $(1.04)^2 \cdot (1 + f_{[2,5]})^3 = (1.05)^5 \Rightarrow f_{[2,5]} = 5.672\%$.

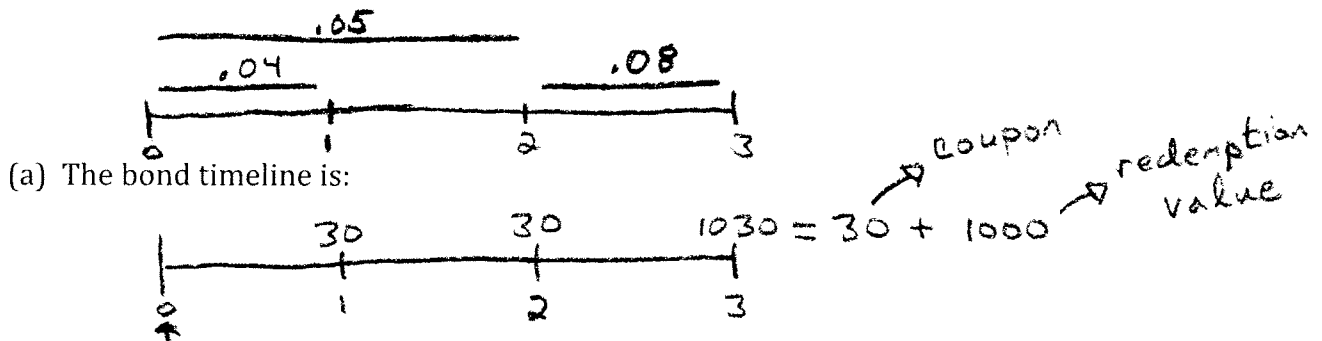
2. The timeline is:



First, note that we cannot talk about *the* bi-annual discount factor because it depends on which 2-year period we're discounting through. The easiest way to determine the bi-annual discount factor from time 2 to time 0 is to take the product of the annual discount factor from time 2 to time 1 and the annual discount factor from time 1 to time 0. That is, the bi-annual discount factor from time 2 to time 0 is $v_{.02} \cdot v_{.03} = \frac{1}{1.02} \cdot \frac{1}{1.03} = \frac{1}{1.0506}$.

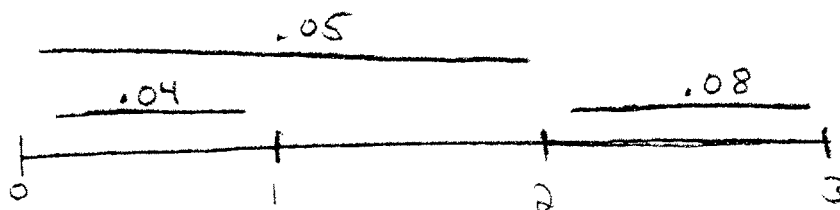
Also notice that if instead of the given information we knew the 2-year spot rate s_2 , then the biannual discount factor from time 2 to time 0 is $v_{s_2}^2 = \frac{1}{(1+s_2)^2}$. In this case, we get s_2 by solving $(1+s_2)^2 = 1.02 \cdot 1.03 = 1.0506$. We actually don't need s_2 but $v_{s_2}^2$, and we get $v_{s_2}^2 = \frac{1}{(1+s_2)^2} = \frac{1}{1.0506}$. This is the same bi-annual discount factor from time 2 to time 0 we found above. The purpose of this paragraph is to show the consistency between the definitions of spot and forward rates.

3. The term structure of interest rates timeline is:



Then $P = 30v_{.04} + 30v_{.05}^2 + 1030v_{.08} \cdot v_{.05}^2 = \frac{30}{1.04} + \frac{30}{1.05^2} + \frac{1030}{1.08 \cdot (1.05)^2} = 921.09$.

(b) The term structure of interest rates timeline, repeated for convenience, is:



In the last problem we found that for a face value of 1000, then the price of the bond is 921.09. Then the annual yield on the bond is determined by solving the equation $921.09 = 30a_{\overline{3}|i} + 1000v_i^3$. Using TVM we get $i \approx 5.95\%$.

We can illustrate the fact that this problem will not depend on the face value of the bond by redoing the problem with a face value of 10,000. The coupons, the redemption value, and the price of the bond with face value of 10,000 are all multiples of 10 of the coupons, the redemption value, and the price of the bond with a face value of 1000. Then we get an equivalent equation to solve; namely, the equation obtained by multiplying the above equation by 10. Therefore we get the same value, $i \approx 5.95\%$ as above. Generally, if the problem does not state a face value, and does not ask for you to determine the face value, then the answer will not depend on the face value and you may use any arbitrary value that you want for it.

4. The annual forward rate from time 1 to 3 is related to the 1-year and 3-year spot rates by

$$(1 + s_3)^3 = (1 + s_1) \cdot (1 + f_{[1,3]})^2$$

The spot rates are $s_1 = i_1 = 0.035$ and $s_3 = i_3 = 0.045$, and so

$$f_{[1,3]} = \sqrt{\frac{1.045^3}{1.035}} - 1 \approx 0.05$$

Section 5: Duration

The **Macauley duration** (or just **duration**) of a sequence of future payments is a measure of the timing of the payments. It is a weighted average of the timing of the payments, where the weight given to the payment time t is equal to the ratio of the *present value* of the payment at time t to the total *present value* of all payments. Since the weights are ratios of present values, then the duration will depend on the interest rate used to discount the payments. I.e., the duration is a function of the interest rate.

If payments of X and Y are made at times k and n , respectively, then the (Macauley) duration of this cash flow is $MacD = \frac{Xv^k}{Xv^k + Yv^n} \cdot k + \frac{Yv^n}{Xv^k + Yv^n} \cdot n$, which can be simplified to $MacD = \frac{kXv^k + nYv^n}{Xv^k + Yv^n}$. This last formula can be generalized to an arbitrary number of payments as follows, letting R_t denote the amount of the payment at time t :

$$MacD = \frac{\sum t \cdot R_t \cdot v^t}{\sum R_t \cdot v^t}$$

Note that the denominator is just the present value of the payments. E.g., if the payments are the coupons and redemption value of a bond, then the denominator is just the price of the bond. Sometimes instead of price, we say the **present value function**, which we denote by $P(i)$, emphasizing the fact that it depends on the interest rate.

Note that since $P(i) = \sum R_t \cdot (1+i)^{-t}$, then using the power rule for derivatives, we have $P'(i) = \sum (-t) \cdot R_t \cdot (1+i)^{-t-1} = -\sum t \cdot R_t \cdot v^{t+1}$. Then, factoring out a v from the summand, we have $P'(i) = -v \sum t \cdot R_t \cdot v^t$. The summation in this last expression is the numerator of the defining expression for $MacD$. This fact is one justification the following definition.

The **modified duration** (or **volatility**) of a sequence of future payments is

$$ModD = -\frac{P'(i)}{P(i)} = v \cdot MacD$$

Remark: Since the price is a decreasing function of the interest rate, the negative sign in the above formula ensures that $ModD$ is positive. Also, for small changes in the interest rate, Δi , then $P(i) \cdot ModD \cdot \Delta i$ will approximate the corresponding change in the price. Therefore, $ModD$ is a measure of how sensitive the present value of the future cash flow is to small changes in the interest rate.

Module 4 Section 5 Problems:

1. Determine the duration of a 10-year zero coupon bond, redeemable at 1000, using an annual effective interest rate of 5%.

Note: Think about it. Look back at the definition of duration, and you should be able to answer this problem without putting pencil to paper.

2. You are given a sequence of 2 payments; 1000 at the end of 2 years and 3000 at the end of 6 years.

(a) Determine the modified duration, using an annual effective interest rate of 4%.

(b) Determine the time value calculated by the method of equated time. The method of equated time is the special case of duration, using a 0% interest rate.

3. Determine the duration of 20-year 6% annual coupon bonds using

(a) an annual effective interest rate of 8%.

(b) an annual effective interest rate of 6%.

Note: Assume the bond is redeemable at par. The face value is not given, and in fact, we will show that for bonds redeemable at par, the duration is independent of the face value. Therefore you may use any face value you want.

4. Determine the duration of a perpetuity-immediate with annual payments of K , using an annual effective interest rate of 8%.
5. Determine the modified duration (in years) of a perpetuity-due with monthly payments of K , using a nominal interest rate of 6% compounded monthly.
6. A stream of periodic payments has a present value of 5000 and a modified duration of 5, both using a periodic effective interest rate of 5%. Approximate the present value of the payments using a periodic effective interest rate of 5.02%.

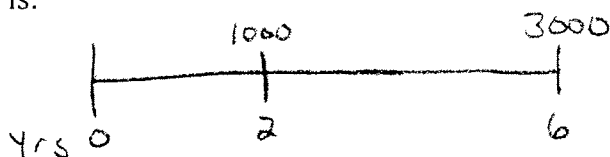
Solutions to Module 4 Section 5 Problems:

1. The timeline is:



Technically we get $MacD = \frac{10 \cdot 1000 \cdot v_{.05}^{10}}{1000 \cdot v_{.05}^{10}} = 10$. Since there is only one payment at time 10 and the (Macaulay) duration is the weighted average of the timing of the payments, then it should be clear that the duration is 10. Also, note that the duration of a zero coupon bond will not depend on the redemption value or the interest rate used in the calculation.

2. The timeline is:

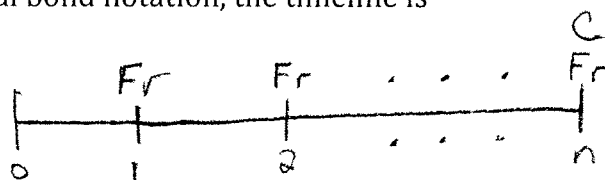


$$MacD = \frac{2 \cdot 1000 \cdot v^2 + 6 \cdot 3000 \cdot v^6}{1000v^2 + 3000v^6}$$

(a) Evaluating at $i = 0.04$ we get $MacD \approx 4.8778$. Then $ModD = v_{.04} \cdot MacD \approx 4.69$.

(b) Evaluating at $i = 0$ ($v = 1$) we get $\bar{t} = \frac{2 \cdot 1000 + 6 \cdot 3000}{1000 + 3000} = 5$.

3. Using general bond notation, the timeline is



$$MacD = \frac{Fr \cdot v + 2(Fr)v^2 + \dots + n(Fr)v^n + nCv^n}{Fr \cdot v + Fr \cdot v^2 + \dots + Fr \cdot v^n + Cv^n}$$

If we factor out Fr from the first n terms of the numerator, the resulting factor is $v + 2v^2 + \dots + nv^n \stackrel{VEP}{=} (Ia)_{\overline{n}|}$. Therefore, for general bonds,

$$MacD = \frac{Fr \cdot (Ia)_{\overline{n}|} + nCv^n}{P}$$

where $P = Price = Fra_{\overline{n}|} + Cv^n$.

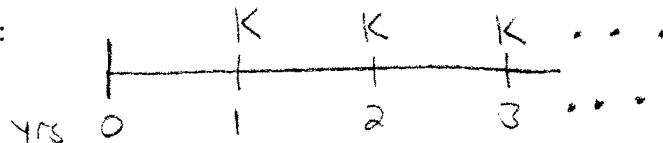
If the bond is redeemable at par, i.e. $F = C$, then every term in the expression for $MacD$ has a factor of F , and so we can cancel it off. We conclude that for bonds redeemable at par, the duration will not depend on the face value. We can therefore use any arbitrary value for the face value, or we can cancel off F and use the formula

$$MacD = \frac{r \cdot (Ia)_{\overline{n}|} + nv^n}{ra_{\overline{n}|} + v^n}$$

(a) Evaluating using $r = 0.06$, $i = 0.08$, and $n = 20$, we get $MacD \approx 11.231$.

(b) Notice that the interest rate we're using for the duration calculation equals the coupon rate ($r = i = 0.06$). Since the bond is redeemable at par, we use the above formula. The numerator reduces to $i \cdot (Ia)_{\overline{n}|} + nv^n = i \cdot \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} + nv^n = \ddot{a}_{\overline{n}|}$ after cancelling the i 's in the first term and adding out nv^n . Similarly, the denominator reduces to $ia_{\overline{n}|} + v^n = i \cdot \frac{1-v^n}{i} + v^n = 1$. Therefore for bonds that are redeemable at par, and using an interest rate i equal to the coupon rate r , we get $MacD = \ddot{a}_{\overline{n}|}$. In this problem, with $n = 20$ and $i = 0.06$, we have $MacD = \ddot{a}_{\overline{20}|0.06} \approx 12.158$.

4. The timeline is:



$$MacD = \frac{Kv + 2Kv^2 + 3Kv^3 \dots}{Kv + Kv^2 + Kv^3 \dots} = \frac{K(v + 2v^2 + 3v^3 \dots)}{K(v + v^2 + v^3 \dots)} = \frac{v + 2v^2 + 3v^3 \dots}{v + v^2 + v^3 \dots}$$

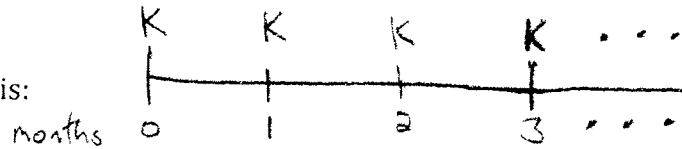
Note that the duration does not depend on the amount, K , of the level payments. Recognize the numerator as the present value of an arithmetically increasing perpetuity-immediate with $P = Q = 1$, and so the numerator equals $\frac{P}{i} + \frac{Q}{i^2} = \frac{1}{i} + \frac{1}{i^2}$. Recognize the denominator as the present value of a perpetuity-immediate with level payments of 1, and so the denominator equals $\frac{1}{i}$. Therefore we have

$$MacD = \frac{\frac{1}{i} + \frac{1}{i^2}}{\frac{1}{i}} = 1 + \frac{1}{i}$$

With $i = 0.08$, we get $MacD = 13.5$.

5.

The timeline is:



We first determine the Macaulay duration in months, and then multiply by the monthly discount factor $v = \frac{1}{1+i} = \frac{1}{1.005}$ to get the modified duration in months. We then divide by 12 to get the modified duration in years. Similar to the last problem,

$$MacD = \frac{Kv + 2Kv^2 + 3Kv^3 \dots}{K + Kv + Kv^2 + Kv^3 \dots} = \frac{K(v + 2v^2 + 3v^3 \dots)}{K(1 + v + v^2 + v^3 \dots)} = \frac{v + 2v^2 + 3v^3 \dots}{1 + v + v^2 + v^3 \dots}$$

As in the last problem, recognize the numerator as the present value of an arithmetically increasing perpetuity immediate with $P = Q = 1$, and so the numerator equals $\frac{P}{i} + \frac{Q}{i^2} = \frac{1}{i} + \frac{1}{i^2}$. This time the denominator is the present value of a perpetuity-due with level payments of 1, and so the denominator equals $\frac{1}{i}(1 + i)$. Therefore we have

$$MacD = \frac{\frac{1}{i} + \frac{1}{i^2}}{\frac{1}{i}(1 + i)} = \frac{1}{i}$$

With $i = 0.005$, we get $MacD = \frac{1}{.005} = 200$ months. Therefore $ModD = 200v_{.005}$ months, and so $ModD = \frac{200v}{12} \approx 16.58$ years.

6.

There is an inverse relationship between the interest rate and the present value of a stream of payments, meaning if the interest rate increases, the present value of the payments decrease. We approximate the change in the present value by evaluating the product $P(i) \cdot ModD \cdot \Delta i$. In this problem, we get that the change in the present value of the payments using a periodic eir of 5.02% rather than 5% is approximately $(5000)(5)(0.0502 - 0.05) = 5$. Since the $5.02 > 5$, we conclude that the present value of the payments decreases by approximately 5. Therefore, the present value of the payments using a periodic eir of 5.02% is approximately 4995.

Section 6: Asset-Liability Management (Immunization and Dedication)

Asset-liability management refers to the process of investing money (assets) and using the return on the investment to pay future obligations (liabilities). Since we're dealing with both assets and liabilities, we have a present value function for assets, $P^A(i) = \sum A_t \cdot v^t$, and a present value function for liabilities, $P^L(i) = \sum L_t \cdot v^t$, where A_t and L_t are the amounts of the asset and liability at time t . The **net present value function** is $P(i) = P^A(i) - P^L(i) = \sum R_t \cdot v^t$ where $R_t = A_t - L_t$ is the net value of the payment at time t ("assets - liabilities"). We study 2 methods of asset-liability management.

Method 1: (Redington and Full Immunization)

We perform immunization at a certain interest rate, i_0 . The idea behind **Redington Immunization**, or just **Immunization**, is to structure assets in such a way that the net present value function has a local minimum of 0 at i_0 . Recall from calculus that this means three things:

1. $P(i_0) = 0$,
2. $P'(i_0) = 0$, and
3. $P''(i_0) > 0$.

Remarks about these three statements:

1. From the definition of the net present value function, the first statement says the present value of the assets equals the present value of the liabilities at interest rate i_0 .
2. Likewise the second statement implies that the derivative of the present value function of the assets equals the derivative of the present value function of the liabilities at interest rate i_0 . Recalling the definition of modified duration, $ModD = -\frac{P'(i)}{P(i)}$, statements 1 and 2 taken together imply that the modified duration of the assets equals the modified duration of the liabilities at interest rate i_0 . However, since $ModD = v \cdot MacD$, the same can be said about the duration; i.e. the duration of the assets equals the duration of the liabilities at interest rate i_0 .
3. It makes sense to introduce terminology regarding the second derivative of the function $P(i_0)$. Similar to the definition of modified duration (volatility), we define the **convexity**, C , of a sequence of future payments as

$$C = \frac{P''(i)}{P(i)}$$

Then statements 1 and 3 taken together imply that the convexity of the assets is *greater than* the convexity of the liabilities.

By the definition of immunization at the interest rate i_0 , the net present value function (pv of assets – pv of liabilities) has a local minimum of 0 at i_0 . Therefore, if we evaluate the net present value function at another interest rate very close to, but not equal to i_0 , then we have a positive net present value; i.e. the present value of the assets is more than the present value of the liabilities. This is true for any small change in the interest rate, in either direction away from i_0 . Wow, this is nice, but ...

In reality, there may be no way to achieve immunization. In theory, if we assume a **flat yield curve**, meaning all the spot rates are equal, then it is possible to achieve immunization. In fact, if for each of the liabilities, we invest in an asset that has two payouts, one before the liability is due and one after the liability is due, then we will have created a net present value function that not only has a local minimum of 0 at i_0 , but has a *global* minimum of 0 at i_0 . Such an arrangement of assets versus liabilities creates what is called **full immunization**. So full immunization at i_0 is a technique to structure assets versus liabilities in a manner that would eliminate the risk of adverse effects created by *all* changes in interest rates away from i_0 , whereas (Redington) immunization at i_0 would eliminate the risk of adverse effects created by *small* changes in interest rates away from i_0 .

Method 2: Absolute or Exact Matching (also called Dedication)

The idea here is that for given liabilities, invest in assets in such a way that when a liability is due, then the assets return an amount exactly equal to the amount of the liability. Then the net value of the payment at each time is 0, i.e. $R_t = 0$, and so the net present value function is identically zero, i.e. $P(i) \equiv 0$, for any interest rate i . Conditions 1 and 2 of immunization are satisfied, but note that condition 3 is *not* satisfied since in this case both the convexity of the assets and liabilities is 0. This method is not an immunization strategy.

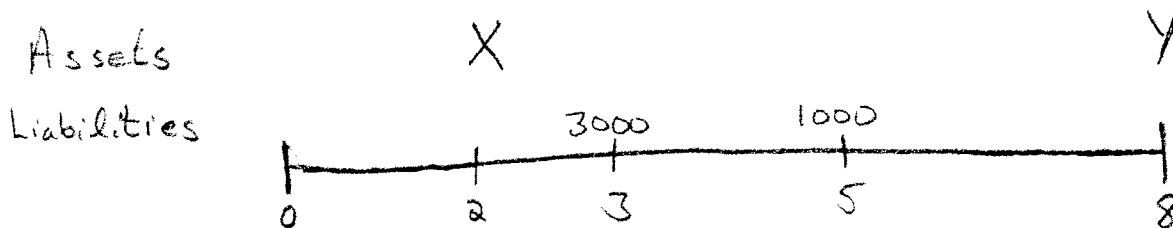
Module 4 Section 6 Problems:

1. Liabilities at times 3 and 5 of amounts 3000 and 1000, respectively, are to be immunized with 2-year and 8-year zero coupon bonds. Assume a flat yield curve of 3%.
 - (a) Determine the cost to immunize the liabilities.
 - (b) Determine the asset amounts at time 2 and at time 8 that are needed in order for the first two conditions of immunization to be satisfied.
 - (c) If the zero coupon bonds are 1000 face value bonds, redeemable at par, then how many of each type of bond should be purchased in order to immunize the liabilities. (Assume we can buy any number (even fractions) of bonds.)
 - (d) Given immunization as above, determine the excess of the present value of assets over the present value of liabilities if the interest rate is changed to 6% annual effective, and then determine the excess of the present value of assets over the present value of liabilities if the interest rate is changed to 1% annual effective.
2. Liabilities of 10,000 in one year and 15,000 in three years are to be exactly matched using 1-year and 3-year zero coupon bonds. Determine the cost to do so if the current 1-year spot rate is 3% and the forward rate from time 1 to time 3 that is consistent with the current term structure of interest rates is 3.5%.
3. A 2-year loan has a required payment of 14,420 in one year and a required payment of 15,900 in two years. These payments are to be exactly matched by investing in 100 face-value 1-year bonds with a 4% annual coupon, and 10,000 face-value 2-year bonds with 6% annual coupons. All bonds are redeemable at par.
 - (a) Determine the number of each type of bond that is needed.
 - (b) Determine the price to exactly match the payments if both bonds can be bought to yield 4% annual effective.
 - (c) Determine the price to exactly match the payments if the 1-year bond can be bought to yield 4% annual effective and the 2-year bond can be bought to yield 5% annual effective.

Solutions to Module 4 Section 6 Problems:

1. (a) The cost to immunize the liabilities can be thought of as the present value of the investments (assets) that are needed to create immunization. However, the first condition for immunization is that the present value of the assets equals the present value of the liabilities. Therefore we can get the cost to immunize by determining the present value of the liabilities. Using a 3% flat yield curve, we have that the cost is $P = 3000v_{.03}^3 + 1000v_{.03}^5 = 3608.03$.

(b) The first two conditions of immunization are satisfied exactly when the present value of assets equals the present value of liabilities and the duration of assets equals the duration of liabilities. The timeline is



Equating present values, we have $Xv^2 + Yv^8 = 3000v^3 + 1000v^5$. Notice that these expressions are the respective denominators for the duration formulas for the assets and liabilities. Since these denominators are equal, then equating durations is equivalent to saying the respective numerators for the duration formulas for assets and liabilities are equal. We get $2Xv^2 + 8Yv^8 = 3(3000)v^3 + 5(1000)v^5$.

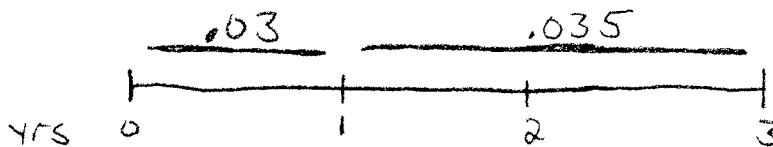
We solve this system of two equations and two unknowns simultaneously. The elimination method is the easiest method to solve the system. For example, multiplying the first equation by negative 2 and then adding to the second equation eliminates X . Using $i = 3\%$, we get $X = 2884.76$ and $Y = 1126.00$.

Remark: The first two conditions of immunization imply the net present value function has a critical point at $i_0 = 3\%$. At first glance, we don't know whether this critical point gives rise to a local minimum (Redington Immunization) or global minimum (Full Immunization), or a local or global maximum (which would imply we may not have enough assets to pay the liabilities), or even a saddle point. However, because in this problem, there is an asset before the first liability and another after the last liability, then the asset values in this case will fully immunize the liabilities at 3%.

- (c) Since we need 2884.76 at time 2 and 1126.00 at time 8, and since the bonds are redeemable at 1000, then we need 2.88476 of the 2-year zero-coupon bonds and 1.126 of the 8-year zero-coupon bonds.

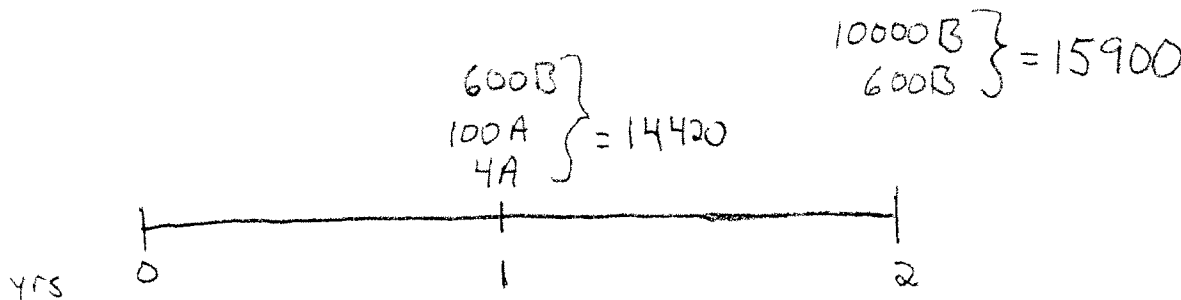
(d) We have assets of 2884.76 and 1126.00 at times 2 and 8, respectively, and we have liabilities of 3000 and 1000 at times 3 and 5, respectively. The excess of the present value of assets over the present value of liabilities is the net present value; $NPV = P(i) = (2884.76v^2 + 1126v^8) - (3000v^3 + 1000v^5)$. Note that evaluating at $i = 3\%$ we get $NPV = P(0.03) = 0$, as expected since we're immunized at $i = 3\%$. Evaluating at $i = 6\%$ we get $NPV = P(0.06) = 7.78$. This means that if interest rates changed to 6%, then the present value of assets would be 7.78 more than the present value of liabilities. On the other hand, evaluating at $i = 1\%$ we get $NPV = P(0.01) = 4.52$. This means that if interest rates changed to 1%, then the present value of assets would be 4.52 more than the present value of liabilities.

2. In order to exactly match the liabilities, we must have assets of 10,000 in one year and 15,000 in three years. The cost of exactly matching the liabilities is then the present value of these amounts. The timeline for interest rates is:



Therefore the cost is $P = 10000v_{.03} + 15000v_{.035}^2v_{.03} = 23,303.55$.

3. (a) Let A denote the number of 1-year bonds purchased, and let B denote the number of 2-year bonds purchased. For each of the 1-year bonds, there is a coupon of $Fr = 100(.04) = 4$ and a redemption value of 100. Therefore, if A of these bonds is purchased, then there is a total of $4A$ paid in coupons and $100A$ in redemption value. Likewise, for each of the 2-year bonds, there is a coupon at the end of each year of $Fr = 10000(.06) = 600$ and a redemption value of 10000. Therefore, if B of these bonds is purchased, then there is a total of $600B$ paid in coupons at the end of each year and $10000B$ in redemption value. Therefore the timeline is:



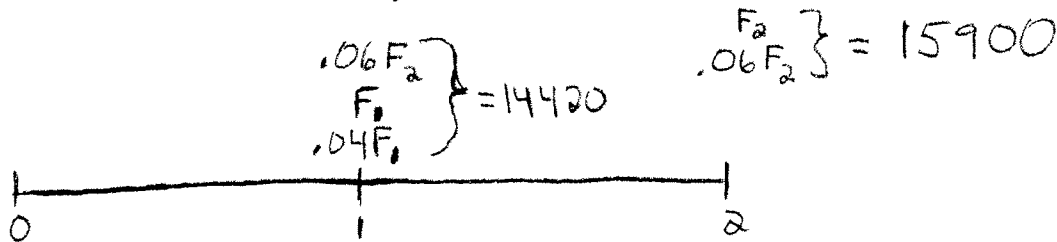
Exact matching implies $104A + 600B = 14420$ and $10600B = 15900$, which implies $A = 130$ and $B = 1.5$. Therefore we need to purchase 130 of the 100 face value 4% annual coupon 1-year bonds and 1.5 of the 10,000 face value 6% annual coupon 2-year bonds.

Remark: Mathematically, buying n bonds with a face value of F (redeemable at par) is equivalent to buying 1 bond with a face value of $n \cdot F$, since in each case the total coupon amount will be $n \cdot Fr$ and the total redemption value is $n \cdot F$. For example, we found above that we needed to purchase $n = 130$ of the 1-year bonds with a face value of 100 and 4% annual coupon. In this case, we would receive a total in coupons of $130(4) = 520$ at the end of the year, and a total in redemption values of $130(100) = 13,000$ at the end of the year. Instead, we could say that we need to purchase 1 of the 1-year bond with face value of $130(100) = 13,000$ and 4% annual coupon. Once again, in this case we would receive a coupon of $13000(.04) = 520$ at the end of the year, and a redemption value of 13,000 at the end of the year.

Some exam questions may ask for the face amount of a bond needed for a certain situation to occur. For example, this question could be worded as:

"Liabilities of 14,420 in one year and 15,900 in two years are to be exactly matched using a 1-year bond with a 4% annual coupon and a 2-year bonds with 6% annual coupons. Both bonds are redeemable at par. Determine the face amount of each type of bond."

In reality, a bond cannot be bought with an arbitrary face value, but because of the above equivalence, we can assume an arbitrary face value. With this problem we may proceed as follows: let F_1 denote the face amount of the 1-year bond and F_2 denote the face amount of the 2-year bond. Then the timeline is:



Exact matching implies $1.04F_1 + .06F_2 = 14420$ and $1.06F_2 = 15900$. Then $F_1 = 13000$ and $F_2 = 15000$, which is equivalent to our answer above.

(b) Since both bonds are bought to yield the same rate, 4% annual effective, then we can determine the price to exactly match the payments by discounting the payments at 4% annual effective. So the cost is $P = 14420v_{.04} + 15900v_{.04}^2 = 28,565.83$.

Alternatively, we can use our work in part (a). The more realistic approach is that we need to buy 130 of the 100 face value 4% annual coupon 1-year bonds and 1.5 of the 10,000 face value 6% annual coupon 2-year bonds. The 1-year bonds cost $P_1 = 4a_{\overline{1}|.04} + 100v_{.04} = 100$ each, whereas $P_2 = 600a_{\overline{2}|.04} + 10000v_{.04}^2 = 10377.22$ is the cost of each 2-year bond. With 130 1-year bonds and 1.5 2-year bonds, we have a total cost of $130(100) + 1.5(10377.22) = 28,565.83$ as above.

We also found that we could buy 1 of each type of bond by changing face values to $F_1 = 13000$ and $F_2 = 15000$. Then $P_1 = 520a_{\overline{1}|.04} + 13000v_{.04} = 13000$ and $P_2 = 900a_{\overline{2}|.04} + 15000v_{.04}^2 = 15,565.83$, again giving a total cost of 28,565.83.

(c) Since the yield on the 1-year bond is different than the yield on the 2-year bond, we can not use the first method in the first paragraph of the solution to part (b) above. We must use the alternative methods.

The more realistic approach is that we need to buy 130 of the 100 face value 4% annual coupon 1-year bonds and 1.5 of the 10,000 face value 6% annual coupon 2-year bonds. Since the 1-year bonds are bought to yield 4% annual effective, each one costs $P_1 = 4a_{\overline{1}|.04} + 100v_{.04} = 100$. Since each of the 2-year bonds are bought to yield 5% annual effective, $P_2 = 600a_{\overline{2}|.05} + 10000v_{.05}^2 = 10185.94$ is the cost of each 2-year bond. With 130 1-year bonds and 1.5 2-year bonds, we have a total cost of $130(100) + 1.5(10185.94) = 28,278.91$.

We also found that we could buy 1 of each type of bond by changing face values to $F_1 = 13000$ and $F_2 = 15000$. Then $P_1 = 520a_{\overline{1}|.04} + 13000v_{.04} = 13000$ and $P_2 = 900a_{\overline{2}|.05} + 15000v_{.05}^2 = 15,278.91$, again giving a total cost of 28,278.91.